Note: In this problem set, expressions in green cells match corresponding expressions in the text answers.

2 - 8 Double integrals

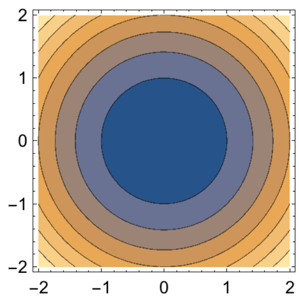
Describe the region of integration and evaluate.

3. $\int_0^3 \int_{-y}^{y} \left(x^2 + y^2\right) \, dx \, dy$

In[56]:=
```
ClearAll["Global`*"]
```

In[57]:=
```
ContourPlot[{x^2 + y^2}, {x, -2, 2}, {y, -2, 2}, ImageSize → 150]
```

Out[57]=



This is a circle in 2d, in 3d a pothole.

In[58]:=
```
e1 = Integrate[Integrate[x^2 + y^2, {x, -y, y}], {y, 0, 3}]
```

Out[58]= 54

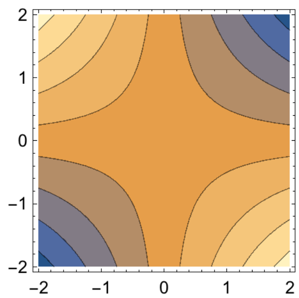
I don't have the intermediate answer for the above, Mathematica just performed the double integration in one step.

5. $\int_0^1 \int_{x^2}^{x} (1 - 2\,x\,y) \, dy \, dx$

In[59]:=
```
ClearAll["Global`*"]
```

In[60]:=
```
e1 = ContourPlot[(1 - 2 x y), {x, -2, 2}, {y, -2, 2}, ImageSize → 150]
```

Out[60]=



This is a saddle surface.

In[61]:= e2 = Integrate[Integrate[(1 - 2 x y), {y, x^2, x}], {x, 0, 1}]

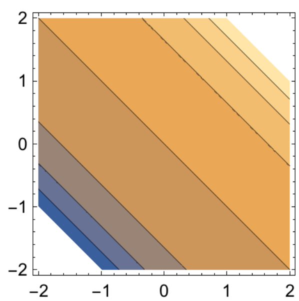
Out[61]= $\frac{1}{12}$

6. $\int_0^2 \int_0^y \text{Sinh}[x + y] \, dx \, dy$

7. $\int_2^0 \int_y^0 \text{Sinh}[x + y] \, dx \, dy$ (problem 6, order reversed)

In[62]:= ClearAll["Global`*"]

In[63]:= e1 = ContourPlot[Sinh[x + y], {x, -2, 2}, {y, -2, 2}, ImageSize → 150]

Out[63]=


This is a beveled corner, but in 2d, can't tell whether inside of corner is up or down.

In[64]:= e3 = Integrate[Integrate[(Sinh[x + y]), {x, y, 0}], {y, 2, 0}] // Simplify

Out[64]= (-1 + Cosh[2]) Sinh[2]

In[65]:= e4 = PossibleZeroQ[(-1 + Cosh[2]) Sinh[2] - ($\frac{1}{2}$ Sinh[4] - Sinh[2])]

Out[65]= True

The answer above is equivalent to the text answer, as shown by the PZQ. I did not understand what was meant by 'order reversed', I thought it referred to the order of integration, not the order of integration limits.

9 - 11 Volume

Find the volume of the given region in space.

9. The region beneath $z = 4 x^2 + 9 y^2$ and above the rectangle with vertices {0,0},{3,0},{3,2},{0,2}

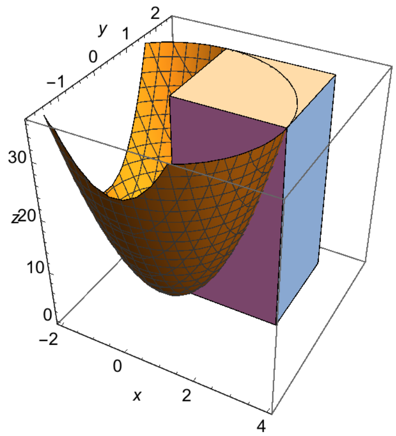
In[66]:= ClearAll["Global`*"]

In[67]:= e1 = ContourPlot3D[z == 4 x^2 + 9 y^2, {x, -2, 4}, {y, -1.75, 2.5},
{z, 0, 36}, AxesLabel → {x, y, z}, ImageSize → 200];

In[68]:=
```
e3 = Graphics3D[Hexahedron[{{0, 0, 0}, {3, 0, 0}, {3, 2, 0},
     {0, 2, 0}, {0, 0, 36}, {3, 0, 36}, {3, 2, 36}, {0, 2, 36}}]];
```

In[69]:=
```
Show[e1, e3]
```

Out[69]=



In[70]:=
```
wib = Integrate[Integrate[4 x^2 + 9 y^2, {x, 0, 3}], {y, 0, 2}]
```

Out[70]= 144

In[71]:= $\int_0^2 \int_0^3 (4x^2 + 9y^2)\, dx\, dy$

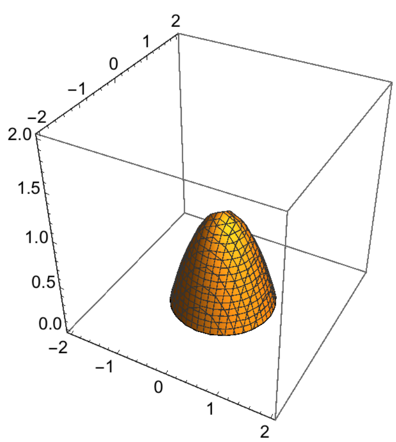
Out[71]= 144

11. The region above the xy-plane and below the paraboloid $z = 1 - (x^2 + y^2)$.

In[72]:=
```
ClearAll["Global`*"]
```

In[73]:=
```
e1 = ContourPlot3D[z == 1 - (x^2 + y^2),
  {x, -2, 2}, {y, -2, 2}, {z, 0, 2}, ImageSize → 200]
```

Out[73]=

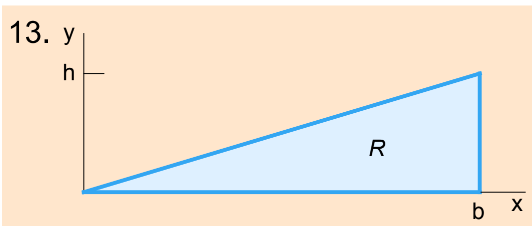


In[74]:=
```
e2 = Solve[(x^2 + y^2) == 1, x]
```
Out[74]= $\{\{x \to -\sqrt{1-y^2}\}, \{x \to \sqrt{1-y^2}\}\}$

In[75]:=
```
e3 = Integrate[Integrate[1 - (x^2 + y^2), {x, -Sqrt[1 - y^2], Sqrt[1 - y^2]}], {y, -1, 1}]
```
Out[75]= $\frac{\pi}{2}$

12 - 16 Center of gravity

Find the center of gravity $\{\bar{x}, \bar{y}\}$ of a mass of density f(x, y) = 1 in the given region R.

Out[76]= 

In[77]:=
```
ClearAll["Global`*"]
```

To find the total mass given the density, I need to integrate the function over the region. But I can't just use the width and height, which would double the true result.

In[78]:=
```
e1M = 1/2 Integrate[Integrate[1, {x, 0, b}], {y, 0, h}]
```
Out[78]= $\frac{b\,h}{2}$

This problem is covered in s.m., where the method of deducing the y-function is discussed. What is explained is that the y-function needs to follow the hypoteneuse, so $y = \frac{h x}{b}$.

In[79]:=
```
x̄ = 1/e1M Integrate[Integrate[x 1, {y, 0, h x/b}], {x, 0, b}]
```
Out[79]= $\frac{2\,b}{3}$

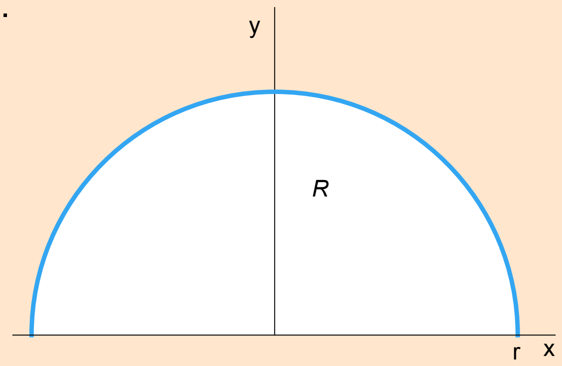
This gives the x coordinate. For the y, it is the same setup, with the only difference being the y as the integrand of the inside integral.

In[80]:=
```
ȳ = 1/e1M Integrate[Integrate[y 1, {y, 0, h x/b}], {x, 0, b}]
```
Out[80]= $\frac{h}{3}$

Note: Mathematica version 10 has a function which calculates the answer automatically. It is called **RegionCentroid.** (See problem 15).

15.

Out[81]=



In[82]:= **disk = Disk[{0, 0}, r, {0, π}]**

Out[82]= **Disk[{0, 0}, r, {0, π}]**

In[83]:= **RegionCentroid[disk]**

Out[83]= $\{0, \frac{4 r}{3 \pi}\}$

17 - 20 Moments of inertia

Find I_x, I_y, I_0 of a mass of density f[x,y] = 1 in the region R in the figures, which the engineer is likely to need, along with other profiles listed in engineering handbooks.

17. R as in problem 13.

The second moment of inertia can be calculated from axes through the centroid, but with a triangle it should be an easy matter. I used the formula for moments of inertia on text p. 429.

In[84]:= $\mathcal{R}$ = **ImplicitRegion**$\left[0 \le x \le b \;\&\&\; 0 \le y \le h \;\&\&\; y \le \frac{h}{b} x, \{x, y\}\right]$;

First to calculate I_x as

In[85]:= **Integrate**$\left[y^2, \{x, y\} \in \mathcal{R}\right]$

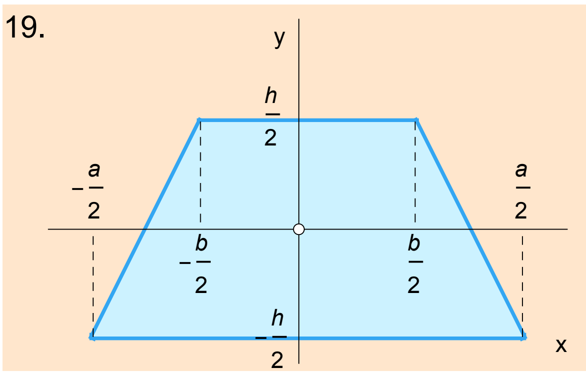
Out[85]= $\begin{cases} 0 & (b > 0 \;\&\&\; h == 0) \;||\; !\;(h \ge 0 \;\&\&\; b > 0) \\ \frac{b h^3}{12} & \text{True} \end{cases}$

Then to calculate I_y as

In[86]:=
```
Integrate[x^2, {x, y} ∈ ℛ]
```

Out[86]=
$$\begin{cases} \frac{b^3}{3} & b > 0 \;\&\&\; h == 0 \\ \frac{b^3 h}{4} & b > 0 \;\&\&\; h > 0 \\ 0 & \text{True} \end{cases}$$

Out[87]=



In[88]:=
```
ClearAll["Global`*"]
```

In[89]:=
$$\frac{\frac{b}{2}}{\frac{a}{2}}$$

Out[89]=
$$\frac{b}{a}$$

The figure is a regular trapazoid (isosceles). I intended to calculate the right side and then double it. So I went about it as

In[90]:=
$$\mathcal{R} = \text{ImplicitRegion}\left[-\frac{h}{2} \le y \le \frac{h}{2} \;\&\&\; 0 \le x \;\&\&\; y \le -\frac{b}{a} x,\ \{x, y\}\right];$$

In[91]:=
```
Integrate[y^2, {x, y} ∈ ℛ, Assumptions → {a, b, h} > 0]
```

Out[91]=
$$\begin{cases} 0 & (a<0 \;\&\&\; b<0 \;\&\&\; h==0) \;||\; (a>0 \;\&\&\; b>0 \;\&\&\; h==0) \;|| \\ & (a<0 \;\&\&\; b==0 \;\&\&\; h==0) \;||\; (a<0 \;\&\&\; b>0 \;\&\&\; h==0) \;|| \\ & (a>0 \;\&\&\; b<0 \;\&\&\; h==0) \;||\; (a>0 \;\&\&\; b==0 \;\&\&\; h==0) \;|| \\ & !\,((h \ge 0 \;\&\&\; a>0) \;||\; (h \ge 0 \;\&\&\; a<0)) \\ \frac{a h^4}{64 b} & (a<0 \;\&\&\; b<0 \;\&\&\; h>0) \;||\; (a>0 \;\&\&\; b>0 \;\&\&\; h>0) \\ \infty\, \text{Sign}[h]^3 & (a<0 \;\&\&\; b==0 \;\&\&\; h>0) \;||\; (a>0 \;\&\&\; b==0 \;\&\&\; h>0) \\ -\frac{5 a h^4}{192 b} + \infty\, \text{Sign}[h]^3 & \text{True} \end{cases}$$

Above: useless. Consider the following grid for I_x

In[92]:= Grid[{{"I_x (1)", "I_x (2)", "I_x (3)"},
$\left\{\frac{h^3 (3 a + b)}{12}, \frac{h^3 (a^2 + 4 a b + b^2)}{36 (a + b)}, \frac{(a + b) h^3}{24}\right\}$}, Frame → All]

Out[92]=

I_x (1)	I_x (2)	I_x (3)
$\frac{1}{12} (3 a + b) h^3$	$\frac{(a^2+4 a b+b^2) h^3}{36 (a+b)}$	$\frac{1}{24} (a + b) h^3$

Or the grid for I_y

In[93]:= Grid[{{"I_y (1)", "I_y (2)", "I_y (3)"},
$\left\{\frac{h (a + b) (a^2 + 7 b^2)}{48}, \frac{h^3 (a + b) (a^2 + b^2)}{48}, \frac{h (a^4 - b^4)}{48 (a - b)}\right\}$}, Frame → All]

Out[93]=

I_y (1)	I_y (2)	I_y (3)
$\frac{1}{48} (a + b) (a^2 + 7 b^2) h$	$\frac{1}{48} (a + b) (a^2 + b^2) h^3$	$\frac{(a^4-b^4) h}{48 (a-b)}$

No agreement seen in these.
Sources for the above. @(1) is *https://www.efunda.com/math/areas/IsosTrapezoid.cfm* (looks like isosceles case)
@(2) is *https://structx.com/Shape_Formulas_015.html* (the page is for isosceles)
@(3) is the text answer for this problem.

Note: In MMA version 10.4 a new function was introduced called **MomentOfInertia**. I don’t have 10.4 but I have 11.1. I tried it out but it don’t return an answer.

To me **MomentOfInertia** is not very impressive. It appears unable to process a point of rotation given in symbolic form.

In[94]:= RegionCentroid[ImplicitRegion[$-\frac{h}{2} \le y \le \frac{h}{2}$ && $0 \le x$ && $y \le -\frac{b}{a} x$, {x, y}]]

Out[94]= ConditionalExpression[
$$\begin{cases} \{0, 0\} & (a < 0 \&\& b < 0 \&\& h = \\ & \quad (a > 0 \&\& b > 0 \&\& h \\ \left\{\frac{a h}{6 b}, -\frac{h}{3}\right\} & (a < 0 \&\& b < 0 \&\& h > \\ & \quad (a > 0 \&\& b > 0 \&\& h \\ \{\text{Indeterminate}, \text{Indeterminate}\} & (a < 0 \&\& b == 0 \&\& h \\ & \quad (a > 0 \&\& b == 0 \&\& h \\ \left\{\frac{\infty}{-\frac{3 a h^2}{8 b}+h\,\infty}, \frac{a h^3}{24 b \left(-\frac{3 a h^2}{8 b}+h\,\infty\right)}\right\} & \text{True} \end{cases}$$
(a > 0 && h > 0) || (a < 0 && b < 0 && h ≥ 0) || (a < 0 && h > 0)]

In[95]:= isotrap = Polygon[$\left\{\left\{-\frac{a}{2}, -\frac{h}{2}\right\}, \left\{\frac{a}{2}, -\frac{h}{2}\right\}, \left\{\frac{b}{2}, \frac{h}{2}\right\}, \left\{-\frac{b}{2}, \frac{h}{2}\right\}\right\}$]

Out[95]= Polygon[$\left\{\left\{-\frac{a}{2}, -\frac{h}{2}\right\}, \left\{\frac{a}{2}, -\frac{h}{2}\right\}, \left\{\frac{b}{2}, \frac{h}{2}\right\}, \left\{-\frac{b}{2}, \frac{h}{2}\right\}\right\}$]

The following seems like a pretty large defect in the program, in that this function also seem incapable of operating on symbol variables.

In[96]:= **RegionCentroid[isotrap]**

RegionCentroid::nmet: Unable to compute the centroid of region Polygon$\left[\left\{\left\{-\frac{a}{2}, -\frac{h}{2}\right\}, \left\{\frac{a}{2}, -\frac{h}{2}\right\}, \left\{\frac{b}{2}, \frac{h}{2}\right\}, \left\{-\frac{b}{2}, \frac{h}{2}\right\}\right\}\right]$. >>

Out[96]= $\text{RegionCentroid}\left[\text{Polygon}\left[\left\{\left\{-\frac{a}{2}, -\frac{h}{2}\right\}, \left\{\frac{a}{2}, -\frac{h}{2}\right\}, \left\{\frac{b}{2}, \frac{h}{2}\right\}, \left\{-\frac{b}{2}, \frac{h}{2}\right\}\right\}\right]\right]$

In[97]:= **isotrapp = Polygon[{{-3, -1.5}, {3, -1.5}, {2, 1.5}, {-2, 1.5}}]**

Out[97]= **Polygon[{{-3, -1.5}, {3, -1.5}, {2, 1.5}, {-2, 1.5}}]**

The defect seems confirmed when specific values are used. By the way, as seen here, it is not necessary to draw the last side of the polygon when looking for its centroid, it is merely required to get all the points listed.

In[98]:= **RegionCentroid[isotrapp]**

Out[98]= **{0., -0.1}**

In[99]:= $\text{d2} = \text{ImplicitRegion}\left[-1.5 \le y \le 1.5 \;\&\&\; y \le -\frac{1.5\,x}{3} \;\&\&\; y \le \frac{1.5\,x}{3}, \{x, y\}\right]$

Out[99]= **ImplicitRegion[-1.5 ≤ y ≤ 1.5 && y ≤ -0.5 x && y ≤ 0.5 x, {x, y}]**

An explicitly described implicit region can also be processed.

In[100]:= **RegionCentroid[d2]**

Out[100]= $\left\{-8.40257\times 10^{-19}, -1.\right\}$